RM-238 AH-2 ES-12
November 1969

ON A METHOD OF SUM COMPOSITION OF ORTHOGONAL LATIN SQUARES*

By

A. Hedayat[1] and E. Seiden

Michigan State University

* This reaearch was supported by NIH GM-05900-11 grant at Cornell University and NSF grant GP-12881 at Michigan State Universtty.

1) On leave from Cornell University.

Abstract

A new method of construction of latin squares and orthogonal latin squares is introduced here. We use a method of sum composition as contrasted with product composition used by other authors. We shall exhibit a method of construction of a latin square of order $n = n_1 + n_2$ given the squares of order n_1 and n_2. Two theorems are proved regarding the construction of a pair of orthogonal squares of order $n = n_1 + n_2$ for $n_1 \geq 7$ except $n_1 = 13$, where $n_1 = p^\alpha$ p an odd prime or $n_1 = 2^\alpha$, α a positive integer provided that $n_2 = (n_1-1)/2$ and $n_2 = n_1/2$ respectively. The construction includes an infinite collection of pairs of orthogonal latin squares of order $4t + 2$. For this it is necessary and sufficient that $p \equiv 7 \pmod 8$ and α odd.

Futther research is in progress indicating possibilities of obtaining more results in two directions. One consists in changing the relative magnitudes of n_1 and n_2, the other in increasing the number of orthogonal squares whenever the values of n_1 and n_2 allow for it.

1. Introduction. Perhaps one of the most useful techinques for the construction of combinatorial systems is the method of composition. To mention some, here are few well-known examples: 1) If there exists a set of t orthogonal latin squares of order n_1 and if there exists a set of t orthogonal latin squares of order n_2, then there exists a set of t orthogonal latin squares of order n_1n_2. 2) If there are Steiner triple systems of order v_1 and v_2, there is a Steiner triple system of order $v = v_1v_2$. 3) If H_1 and H_2 are two Hadamard matrices of order n_1 and n_2 respectively, then the Kronecker product of H_1 and H_2 is a Hadamard matrix of order n_1n_2. 4) If Room squares of order n_1 and n_2 exist, then a Room square of order n_1n_2 exists. 5) If BIB (v_1,k,λ_1) and BIB (v_2,k,λ_2) exist and if $f(\lambda_2v_2^2) \geq k$, then BIB $(v_1v_2,k,\lambda_1\lambda_2)$ exists where $f(\lambda_2v_2^2)$ denotes the maximum number of constraints which are

possible in an orthogonal array of size $\lambda_2 v_2^2$, with v_2 levels, strength 2, and index λ_2. 6) As a final example, the existence of orthogonal arrays $(\lambda_i v_i^t, q_i, v_i, t)$, $i = 1,2,\ldots,r$ implies the existence of the orthogonal array $(\lambda v^t, q, v, t)$, where $\lambda = \lambda_1\lambda_2\cdots\lambda_r$, $v = v_1 v_2 \cdots v_r$, and $q = \min(q_1, q_2, \ldots, q_r)$.

The reader will note that each of the above examples involved a product type composition. The method that we will describe utilizes a sum type composition, by means of which one can possibly construct sets of orthogonal latin squares for all $n \geq 10$.

2. Definitions. In the sequel by an $O(n,t)$ set we mean a set of t mutually orthogonal latin squares of order n.

a) A transversal (directrix) of a latin square L of order n on an n-set Σ is a collection of n cells such that the entries of these cells exhaust the set Σ and every row and column of L is represented in this collection. Two transversals are said to be parallel if they have no cell in common.

b) A collection of n cells is said to form a common transversal for an $O(n,t)$ set if the collection is a transversal for each of these t latin squares. Similarly, two common transversals are said to be parallel if they have no cell in common.

Example. The underlined and paranthesized cells form two parallel common transversals for the following $O(4,2)$ set.

$$\left\{\begin{array}{cccc@{\qquad}cccc}
1 & 2 & (3) & \underline{4} & 1 & 2 & (3) & \underline{4} \\
(2) & \underline{1} & 4 & 3 & (4) & \underline{3} & 2 & 1 \\
\underline{3} & (4) & 1 & 2 & \underline{2} & (1) & 4 & 3 \\
4 & 3 & \underline{2} & (1) & 3 & 4 & \underline{1} & (2)
\end{array}\right\}$$

3. Composing Two Latin Squares of Order n_1 and n_2.

A very natural question in the theory of latin squares is the following: Given two latin squares L_1 and L_2 of order n_1 and n_2 $(n_1 \geq n_2)$ respectively. In how many ways can one compose L_1 and L_2 in order to obtain a latin square L_3 of order m, where m is a function of n_1 and n_2 only? This question can be partially answered as follows. First, it is well-known that the Kronecker product $L_3 = L_1 \otimes L_2$ is a latin square of order $m = n_1 n_2$ irrespective of the combinatorial structure of L_1 and L_2. Secondly, we show that if L_1 has a certain combinatorial structure, then one can construct a latin square L of order $n = n_1 + n_2$. Naturally enough we call this procedure a "method of sum composition".

Even though our method of sum composition does not work for all pairs of latin squares, it has an immediate application in the construction of orthogonal latin squares including those of order $4t + 2$, $t \geq 2$. We emphasize that the combinatorial structure of orthogonal latin squares constructed by the method of sum composition is completely different from those of known orthogonal latin squares in the literature. Therefore, it is worthwhile to study these squares for the purpose of constructing new finite projective planes.

We shall now describe the method of "sum composition". Let L_1 and L_2 be two latin squares of order n_1 and n_2, $n_1 \geq n_2$, on two non-intersecting sets $\Sigma_1 = \{a_1, a_2, \ldots, a_{n_1}\}$ and $\Sigma_2 = \{b_1, b_2, \ldots, b_{n_2}\}$ respectively. If L_1 has n_2 parallel transversals then we can compose L_1 with L_2 to obtain a latin square L of order $n = n_1 + n_2$. Note that for any pair (n_1, n_2), there exists L_1 and L_2 with the above requirement, except for (2,1), (2,2), (6,5) and (6,6).

To produce L put L_1 and L_2 in the upper left and lower right corner respectively. Call the resulting square C_1, which looks as follows:

$$C_1 = \begin{array}{|c|c|} \hline L_1 & \\ \hline & L_2 \\ \hline \end{array}$$

Name the n_2 transversals of L_1 in any manner from 1 to n_2. Now fill the cell $(i, n_1 + k)$, $k = 1,2,\dots,n_2$, with that element of transversal k which appears in row i, $i = 1,2,\dots,n_1$. Fill also the cell $(n_1 + k, j)$, $k = 1,2,\dots,n_2$, with that element of transversal k which appears in column j, $j = 1,2,\dots,n_1$. Call the resulting square C_2. Now every entry of C_2 is occupied with an element either from Σ_1 or Σ_2, but C_2 is obviously not a latin square on $\Sigma_1 \cup \Sigma_2$. However, if we replace each of the n_1 entries of transversal k with b_k, it is easily verified that the resulting square which we call L is a latin square of order n on $\Sigma_1 \cup \Sigma_2$.

The procedure described for filling the first n_1 entries of the row (column) $n_1 + k$ with the corresponding entries of transversal k is, naturally enough, called <u>the projection of transversal k on the first n_1 entries of row (column)</u> $n_1 + k$.

We shall now elucidate the above procedure via an example. Let $\Sigma_1 = \{1,2,3,4,5\}$, $\Sigma_2 = \{6,7,8\}$,

$$L_1 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{matrix} \quad \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad \text{and} \qquad L_2 = \begin{matrix} 6 & 7 & 8 \\ 7 & 8 & 6 \\ 8 & 6 & 7 \end{matrix} \quad .$$

Note that the cells on the same curve in L_1 form a transversal.

$$C_1 = \begin{array}{|ccccc|ccc|} \hline 1 & 2 & 3 & 4 & 5 & & & \\ 5 & 1 & 2 & 3 & 4 & & & \\ 4 & 5 & 1 & 2 & 3 & & & \\ 3 & 4 & 5 & 1 & 2 & & & \\ 2 & 3 & 4 & 5 & 1 & & & \\ \hline & & & & & 6 & 7 & 8 \\ & & & & & 7 & 8 & 6 \\ & & & & & 8 & 6 & 7 \\ \hline \end{array} \qquad \text{and} \qquad C_2 = \begin{array}{|ccccc|ccc|} \hline 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 & 3 & 4 & 5 \\ \hline 1 & 3 & 5 & 2 & 4 & 6 & 7 & 8 \\ 5 & 2 & 4 & 1 & 3 & 7 & 8 & 6 \\ 4 & 1 & 3 & 5 & 2 & 8 & 6 & 7 \\ \hline \end{array}$$

And finally

$$L = \left|\begin{array}{ccccc|ccc}
6 & 7 & 8 & 4 & 5 & 1 & 2 & 3 \\
7 & 8 & 2 & 3 & 6 & 4 & 5 & 1 \\
8 & 5 & 1 & 6 & 7 & 2 & 3 & 4 \\
3 & 4 & 6 & 7 & 8 & 5 & 1 & 2 \\
2 & 6 & 7 & 8 & 1 & 3 & 4 & 5 \\
\hline
1 & 3 & 5 & 2 & 4 & 6 & 7 & 8 \\
5 & 2 & 4 & 1 & 3 & 7 & 8 & 6 \\
4 & 1 & 3 & 5 & 2 & 8 & 6 & 7
\end{array}\right|$$

which is a latin square of order 8 on $\Sigma_1 \cup \Sigma_2 = \{1,2,\ldots,8\}$.

Remark. Note that it is by no means required that the projection of transversals on the rows and columns should have the same ordering. Indeed, for the fixed set of ordered n_2 transversals, we have $n_2!$ choices of projections on columns and $n_2!$ choices of projections on the rows. Hence we can generate at least $(n_2!)^2$ different latin squares of order $n = n_1 + n_2$ composing L_1 and L_2.

4. Construction of $O(n,2)$ Sets by Method of Sum Composition.

In order to construct an $O(n,2)$ set for $n = n_1 + n_2$, we require that $n_1 \geq 2n_2$ and there should exist an $O(n_2,2)$ set, and an $O(n_1,2)$ set with $2n_2$ parallel transversals. It is easy to show that any $n \geq 10$ can be decomposed in at least one way into $n_1 + n_2$ which fulfill the above requirements. We now present two theorems which state that for certain n one can construct an $O(n,2)$ set by the method of sum composition.

Theorem 4.1. Let $n_1 = p^\alpha \geq 7$ for any odd prime p and positive integer α, excluding $n_1 = 13$. Then there exists an $O(n,2)$ set which can be constructed by composition of two $O(n_1,2)$ and $O(n_2,2)$ sets for $n_2 = (n_1-1)/2$ and $n = n_1 + n_2$.

We shall first give the method of construction and then a proof that the constructed set is an $O(n,2)$ set.

Construction. Let $B(r)$ be the $n_1 \times n_1$ square with element $r\alpha_i + \alpha_j$ in its (i,j) cell, α_i, α_j, $0 \neq r$ in $GF(n_1)$, $i,j = 1,2,\ldots,n_1$. Then it is easy to see that $\{B(1), B(x), B(y)\}$, $y = x^{-1}$, $x \neq 1$, is an $O(n_1, 3)$ set. Consider the n_1 cells in $B(1)$ with $\alpha_i + \alpha_j = k$ a fixed element in $GF(n_1)$. Then the corresponding cells in $B(x)$ and $B(y)$ form a common transversal for the set $\{B(x), B(y)\}$. Name this common transversal by k. It is then obvious that two common transversals k_1 and k_2, $k_1 \neq k_2$ are parallel and hence $\{B(x), B(y)\}$ has n_1 common parallel transversals. Now let $\{A_1, A_2\}$ be any $O(n_2, 2)$ set, which always exists, on a set Ω non-intersecting with $GF(n_1)$. For any λ in $GF(n_1)$ we can find $(n_1-1)/2$ pairs of distinct elements belonging to $GF(n_1)$ such that the sum of the two elements of each pair is equal to λ. Let $\{S\}$ and $\{T\}$ denote the collection of the first and the second elements of these $(n_1-1)/2$ pairs respectively. Note that for a fixed λ the set $\{S\}$ can be constructed in $(n_1-1)(n_1-3)\ldots1$ distinct ways. Now fix λ and let L_1 denote any of the $(n_2!)^2$ latin squares that can be generated by the sum composition of $L(x)$ and A_1 using transversals determined by the n_2 elements of $\{S\}$. Let L_2 be the latin square derived from the composition of $L(y)$ and A_2 using the n_2 transversals determined by the elements of $\{T\}$ and the following projection rule: Project transversals t_i, $i = 1,2,\ldots,n_2$ on the row (column) which upon superposition of L_2 on L_1 this row (column) should coincide with the row (column) stemmed from the transversal $\lambda - t_i$. Shortly we shall prove that $\{L_1, L_2\}$ forms an $O(n,2)$ set.

The preceding arguments shows that $\{L_1, L_2\}$ can be constructed non-isomorphically in at least $(n_1-3)(n_2!)^2[n_1(n_1-1)(n_1-3)\ldots1]$ ways. For instance in the case of $n_1 = 7$, there is at least 12096 non-isomorphic pairs of orthogonal latin squares of order 10. Therefore, Euler has been wrong in his conjecture by a very wide margin.

<u>Note that we can construct infinitely many pairs of orthogonal latin squares of order</u> $4t + 2$ <u>by the method of theorem 4.1.</u> <u>For</u> $p \equiv 7 \bmod 8$ <u>and</u> α <u>odd</u> $p^{\alpha} = (8t + 5)/3)$. <u>Hence</u> $n_1 + n_2 = 4t + 2$.

Proof. The constructional procedure clearly reveals that:

A. L_1 and L_2 are latin squares of order n on $GF(n_1) \cup \Omega$.

B. Upon superposition of L_1 on L_2 the following are true:

b_1. Every element of Ω appears with every other element of Ω.

b_2. Every element of Ω appears with every elements of $GF(n_1)$.

b_3. Every element of $GF(n_1)$ appears with every element of Ω.

Therefore, all we have to prove is that every element of $GF(n_1)$ appears with every other element of $GF(n_1)$. To prove this recall that $B(x)$ is orthogonal to $B(y)$. However, since we removed the n_2 transversals from $B(x)$ determined by the n_2 elements of $\{S\}$ and n_2 transversals from $B(y)$ determined by the n_2 elements of $\{T\}$ therefore the following $2n_2n_1$ pairs have been lost.

$$(x\alpha_i + \alpha_j, y\alpha_i + \alpha_j) \text{ with } \alpha_i + \alpha_j = \gamma \text{ for any } \gamma \in GF(n_1), \gamma \neq \lambda,$$

We claim that the given projection rules guarantee the capture of these lost pairs by the $2n_2n_1$ bordered cells. To show this note that the superposition of the projected transversal s from $B(x)$ on the projected transversal $t = \lambda - s$ from $B(y)$ will caputer the n_1 pairs.

$$(x\alpha_i + \alpha_j, y\alpha_i + \alpha_j) \text{ with } \alpha_i + \alpha_j = k = [y(\lambda - s) + s]/(1 + y)$$

if these transversals have been projected on row border and n_1 pairs

$$(x\alpha_i + \alpha_j, y\alpha_i + \alpha_j) \text{ with } \alpha_i + \alpha_j = k = [s(y-1) + (s-\lambda)(x-1)]/(y-x)$$

if these transversals have been projected on column border. Now beeause $k + k' = \lambda$ and if $s_1 \neq s_2$ then $k_1 \neq k_2$ and $k_1' \neq k_2'$ hence the $2n_2n_1$ pairs which have been resulted from the projection of transversals determined by $\{S\}$ and $\{T\}$ will jointly capture the $2n_2n_1$ lost pairs and thus a proof.

We shall now clarify the above constructional procedure by an example.

<u>Example</u>. Let $n_1 = 7$, $GF(7) = \{0,1,2,\ldots,6\}$. Then for $x = 2$, $y = x^{-1} = 4$ we have

$\{B(1),B(2),B(4)\} =$

$$\begin{array}{ccccccc} 0&1&2&3&4&5&6\\ 1&2&3&4&5&6&0\\ 2&3&4&5&6&0&1\\ 3&4&5&6&0&1&2\\ 4&5&6&0&1&2&3\\ 5&6&0&1&2&3&4\\ 6&0&1&2&3&4&5 \end{array} \qquad \begin{array}{ccccccc} 0&1&2&3&4&5&6\\ 2&3&4&5&6&0&1\\ 4&5&6&0&1&2&3\\ 6&0&1&2&3&4&5\\ 1&2&3&4&5&6&0\\ 3&4&5&6&0&1&2\\ 5&6&0&1&2&3&4 \end{array} \qquad \begin{array}{ccccccc} 0&1&2&3&4&5&6\\ 4&5&6&0&1&2&3\\ 1&2&3&4&5&6&0\\ 5&6&0&1&2&3&4\\ 2&3&4&5&6&0&1\\ 6&0&1&2&3&4&5\\ 3&4&5&6&0&1&2 \end{array}$$

For $n_2 = (n_1-1)/2 = 3$ let $\Omega_2 = \{7,8,9\}$ and

$$\{A_1,A_2\} = \begin{array}{ccc} 7&8&9\\ 8&9&7\\ 9&7&8 \end{array}, \begin{array}{ccc} 7&8&9\\ 9&7&8\\ 8&9&7 \end{array}.$$

Finally for $\lambda = 0$, $\{S\} = \{1,2,3\}$ and $\{T\} = \{6,5,4\}$ we have $\{L_1,L_2\} =$

$$\begin{array}{ccccccc|ccc} 0&7&8&9&4&5&6&1&2&3\\ 7&8&9&5&6&0&1&2&3&4\\ 8&9&6&0&1&2&7&3&4&5\\ 9&0&1&2&3&7&8&4&5&6\\ 1&2&3&4&7&8&9&5&6&0\\ 3&4&5&7&8&9&2&6&0&1\\ 5&6&7&8&9&3&4&0&1&2\\ \hline 2&1&0&6&5&4&3&7&8&9\\ 4&3&2&1&0&6&5&8&9&7\\ 6&5&4&3&2&1&0&9&7&8 \end{array} \qquad \begin{array}{ccccccc|ccc} 0&1&2&3&7&8&9&6&5&4\\ 4&5&6&7&8&9&3&2&1&0\\ 1&2&7&8&9&6&0&5&4&3\\ 5&7&8&9&2&3&4&1&0&6\\ 7&8&9&5&6&0&1&4&3&2\\ 8&9&1&2&3&4&7&0&6&5\\ 9&4&5&6&0&7&8&3&2&1\\ \hline 3&0&4&1&5&2&6&7&8&9\\ 6&3&0&4&1&5&2&9&7&8\\ 2&6&3&0&4&1&5&8&9&7 \end{array}$$

the reader can easily verify that $\{L_1,L_2\}$ is an $O(10,2)$ set.

<u>Remarks</u>.

1) The method of theorem 4.1 fails for $n_1 = 13$ only because there is no $O(6,2)$ set. Otherwise, there will be no orthogonality contradiction on the other parts of L_1 and L_2 with their 6×6 lower right square missing.

2) In the case of $n_1 = 7$, if we let $\{S\} = \{0,1,3\}$ and $\{T\} = \{2,4,5\}$ then the requirement $y = x^{-1}$ is not necessary. However then we do not have a unified projection rule for the formation of L_2 as was provided for the case $y = x^{-1}$ by theorem 4.1. To give the complete list of solution let (a_1,a_2,a_3) and (b_1,b_2,b_3) be any two permutations of the set $\{8,9,10\}$. If we project transversals $(0,1,3)$ on the rows

(a_1,a_2,a_3) and columns (b_1,b_2,b_3) in the formation of L_1, then the following table indicates what permutation of transversals $\{2,4,5\}$ should be projected on the rows (a_1,a_2,a_3) and columns (b_1,b_2,b_3) in the formation of L_2. Obviously these permutation will be a function of the pair (x,y).

Pair (x,y)	Rows a_1,a_2,a_3	Columns b_1,b_2,b_3
(2,3)	4, 2, 5	4, 2, 5
(2,3)	2, 5, 4	2, 5, 4
(2,4)	2, 5, 4	4, 2, 5
(2,5)	4, 2, 5	4, 2, 5
(2,6)	2, 5, 4	2, 5, 4
(3,4)	2, 5, 4	2, 5, 4
(3,5)	2, 5, 4	4, 2, 5
(3,5)	4, 2, 5	5, 4, 2
(3,5)	4, 2, 5	2, 5, 4
(3,5)	5, 4, 2	2, 5, 4
(3,6)	4, 2, 5	2, 5, 4
(3,6)	5, 4, 2	4, 2, 5
(4,5)	2, 5, 4	2, 5, 4
(4,6)	5, 4, 2	4, 2, 5
(4,6)	2, 5, 4	2, 5, 4
(4,6)	5, 4, 2	5, 4, 2

(This table is by no means exhaustive.)

The reader may note that whenever $y = x^{-1}$ in the above table the given solution(s) are different from the one provided by the method of theorem 4.1.

Thus we can conclude that any pair of orthogonal latin squares of order 7 based on the GF(7) can be composed with a pair of orthogonal latin squares of order 3 and make a pair of orthogonal latin squares of order 10. In addition, since we have six choices for (a_1,a_2,a_3) and (b_1,b_2,b_3) hence from every line in the above table we can produce 36 non-isomorphic O(10,2) sets or $16 \times 36 = 576$ sets for the entire table. Since all these pairs are non-isomorphic with all previous pairs, produced by theorem 4.1, thus by the method of sum composition one can at least produce 12,672 non-isomorphic $O(10,2)$ sets.

We believe that for other values of n_1 there are sets of $\{S\}$ and $\{T\}$ together with proper projections which makes the restriction $y = x^{-1}$ unnecessary.

Theorem 4.2. Let $n_1 = 2^{\alpha} \geq 8$ for any positive integer α. Then there exists an $O(n,2)$ set which can be constructed by composition of two $O(n_1,2)$ and $O(n_2,2)$ sets for $n_2 = n_1/2$ and $n = n_1 + n_2$.

We shall here give only the method of construction. A similar argument as in theorem 4.1 can be used that the constructed set is an $O(n,2)$ set.

Construction. In a similar fashion as in theorem 4.1 construct the set $\{B(1),B(x),B(y)\}$ over $GF(2^{\alpha})$. Let also $\{A_1,A_2\}$ be any $O(n_2,2)$ set, which always exists, on a set Ω non-intersecting with $GF(2^{\alpha})$. For any $\lambda \neq 0$ in $GF(2^{\alpha})$ we can find $n_1/2$ pairs of distinct elements belonging to $GF(2^{\alpha})$ such that the sum of the two elements of each pair is equal to λ. Let $\{S\}$ and $\{T\}$ denote the collection of the first and the second elements of these $n_1/2$ pairs respectively. Note that for a fixed λ the set $\{S\}$ can be constructed in $n_1(n_1-2(n_1-4)\ldots 1$ distinct ways. Now form L_1 from the sum composition of $B(x)$ and A_1 and L_2 from the sum composition of $B(y)$ and A_2 using the same projection rule as given in theorem 4.1. Now $\{L_1,L_2\}$ is an $O(n,2)$ set.

Example. Let $n = 8$, $GF(8) = \{0,1,2,\ldots,7\}$ with the following addition (+) and multiplication (X) tables:

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	6	4	3	7	2	5
2	2	6	0	7	5	4	1	3
3	3	4	7	0	1	6	5	2
4	4	3	5	1	0	2	7	6
5	5	7	4	6	2	0	3	1
6	6	2	1	5	7	3	0	4
7	7	5	3	2	6	1	4	0

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	3	4	5	6	7	1
3	0	3	4	5	6	7	1	2
4	0	4	5	6	7	1	2	3
5	0	5	6	7	1	2	3	4
6	0	6	7	1	2	3	4	5
7	0	7	1	2	3	4	5	6

Then for $x = 2$, $y = x^{-1} = 7$ we have

$\{B(1), B(2), B(7)\} =$

$$
\begin{array}{cccccccc}
0&1&2&3&4&5&6&7\\
1&0&6&4&3&7&2&5\\
2&6&0&7&5&4&1&3\\
3&4&7&0&1&6&5&2\\
4&3&5&1&0&2&7&6\\
5&7&4&6&2&0&3&1\\
6&2&1&5&7&3&0&4\\
7&5&3&2&6&1&4&0
\end{array}
\qquad
\begin{array}{cccccccc}
0&1&2&3&4&5&6&7\\
2&6&0&7&5&4&1&3\\
3&4&7&0&1&6&5&2\\
4&3&5&1&0&2&7&6\\
5&7&4&6&2&0&3&1\\
6&2&1&5&7&3&0&4\\
7&5&3&2&6&1&4&0\\
1&0&6&4&3&7&2&5
\end{array}
\qquad
\begin{array}{cccccccc}
0&1&2&3&4&5&6&7\\
7&5&3&2&6&1&4&0\\
1&0&6&4&3&7&2&5\\
2&6&0&7&5&4&1&3\\
3&4&7&0&1&6&5&2\\
4&3&5&1&0&2&7&6\\
5&7&4&6&2&0&3&1\\
6&2&1&5&7&3&0&4
\end{array}
$$

For $n_2 = n_1/2 = 4$ let $\Omega = \{A,B,C,D\}$ and

$$
\{A_1, A_2\} = \begin{array}{cccc}
A&B&C&D\\
B&A&D&C\\
C&D&A&B\\
D&C&B&A
\end{array}
,\;
\begin{array}{cccc}
A&B&C&D\\
D&C&B&A\\
B&A&D&C\\
C&D&A&B
\end{array}
\;.
$$

Finally for $\lambda = 5$, $\{S\} = \{0,1,3,4\}$ and $\{T\} = \{5,7,6,2\}$ we have

$\{L_1, L_2\} =$

$$
\begin{array}{cccccccc|cccc}
A&B&2&C&D&5&6&7&0&1&3&4\\
B&A&0&D&C&4&1&3&6&2&5&7\\
3&4&A&0&1&D&B&C&7&5&2&6\\
C&D&5&A&B&2&7&6&1&0&4&3\\
D&C&4&B&A&0&3&1&2&6&7&5\\
6&2&D&5&7&A&C&B&3&4&0&1\\
7&5&B&2&6&C&A&D&4&3&1&0\\
1&0&C&4&3&B&D&A&5&7&6&2\\
\hline
0&6&7&1&2&3&4&5&A&B&C&D\\
2&1&3&6&0&7&5&4&B&A&D&C\\
4&7&6&3&5&1&0&2&C&D&A&B\\
5&3&1&7&4&6&2&0&D&C&B&A
\end{array}
\qquad
\begin{array}{cccccccc|cccc}
0&1&D&3&4&A&C&B&5&7&6&2\\
7&5&C&2&6&B&D&A&0&1&3&4\\
D&C&6&B&A&7&2&5&3&4&0&1\\
2&6&B&7&5&C&A&D&1&0&4&3\\
3&4&A&0&1&D&B&C&7&5&2&6\\
A&B&5&C&D&2&7&6&4&3&1&0\\
C&D&4&A&B&0&3&1&6&2&5&7\\
B&A&1&D&C&3&0&4&2&6&7&5\\
\hline
4&2&7&6&3&5&1&0&A&B&C&D\\
6&3&0&4&2&1&5&7&D&C&B&A\\
5&0&3&1&7&4&6&2&B&A&D&C\\
1&7&2&5&0&6&4&3&C&D&A&B
\end{array}
$$

which is an $O(12,2)$ set.

<u>Discussion</u>. The necessary requriements for the construction of an $O(n,t)$ set, $n = n_1 + n_2$, $t < n_2$, by the method of sum composition are: The existence of an $O(n_1,t)$ set, $n_1 \geq tn_2$, with at least tn_2 common parallel

transversals, and an $O(n_2,t)$ set. These conditions are obviously satisfied whenever n_1 and n_2 are prime powers.

While for some values of n there exists only a unique decomposition fulfilling the above requriements, for infinitely many other values of n there are abundant such decompositions

It seems that if there exists an $O(n_2,2)$ set and if $n = n_1 + n_2$, $n_1 \geq 2n_2$ then one can construct an $O(n,2)$ set by the method of sum composition if n_1 is a prime power. To support this observation and shed some more light on the method of sum composition we present in subsequent pages some highlights of the results which we hope to complete and submit for publication shortly.

In the following for each given decomposition of n we exhibit an $O(n,2)$ set which has been derived by the method of sum composition. We shall represent the pairs in a form that the curious reader can easily reconstruct the original sets. Hereafter the notation $L_1 \perp L_2$ means that L_1 is orthogonal to L_2.

1) 12 = 9 + 3

$$
\begin{array}{ccccccccc|ccc}
A & B & C & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\
B & C & A & 1 & 2 & 3 & 4 & 5 & 6 & 9 & 7 & 8 \\
C & A & B & 7 & 8 & 9 & 1 & 2 & 3 & 5 & 6 & 4 \\
2 & 3 & 1 & 5 & 6 & 4 & A & B & C & 8 & 9 & 7 \\
8 & 9 & 7 & 2 & 3 & 1 & B & C & A & 4 & 5 & 6 \\
5 & 6 & 4 & 8 & 9 & 7 & C & A & B & 3 & 1 & 2 \\
3 & 1 & 2 & A & B & C & 9 & 7 & 8 & 6 & 4 & 5 \\
9 & 7 & 8 & B & C & A & 6 & 4 & 5 & 2 & 3 & 1 \\
6 & 4 & 5 & C & A & B & 3 & 1 & 2 & 7 & 8 & 9 \\
\hline
1 & 5 & 9 & 6 & 7 & 2 & 8 & 3 & 4 & A & B & C \\
7 & 2 & 6 & 3 & 4 & 8 & 5 & 9 & 1 & B & C & A \\
4 & 8 & 3 & 9 & 1 & 5 & 2 & 6 & 7 & C & A & B
\end{array}
\quad \perp \quad
\begin{array}{ccccccccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & A & B & C & 8 & 9 & 7 \\
9 & 7 & 8 & 3 & 1 & 2 & B & C & A & 6 & 4 & 5 \\
5 & 6 & 4 & 8 & 9 & 7 & C & A & B & 1 & 2 & 3 \\
6 & 4 & 5 & A & B & C & 3 & 1 & 2 & 7 & 8 & 9 \\
2 & 3 & 1 & B & C & A & 8 & 9 & 7 & 5 & 6 & 4 \\
7 & 8 & 9 & C & A & B & 4 & 5 & 6 & 3 & 1 & 2 \\
A & B & C & 2 & 3 & 1 & 5 & 6 & 4 & 9 & 7 & 8 \\
B & C & A & 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\
C & A & B & 6 & 4 & 5 & 9 & 7 & 8 & 2 & 3 & 1 \\
\hline
4 & 9 & 2 & 5 & 7 & 3 & 6 & 8 & 1 & A & B & C \\
3 & 5 & 7 & 1 & 6 & 8 & 2 & 4 & 9 & C & A & B \\
8 & 1 & 6 & 9 & 2 & 4 & 7 & 3 & 5 & B & C & A
\end{array}
$$

3) 14 = 11 + 3, the only decomposition which fulfill the necessary requirements.

A	B	C	3	4	5	6	7	8	9	10	0	1	2
B	C	9	10	0	1	2	3	4	5	A	6	7	8
C	4	5	6	7	8	9	10	0	A	B	1	2	3
10	0	1	2	3	4	5	6	A	B	C	7	8	9
6	7	8	9	10	0	1	A	B	C	5	2	3	4
2	3	4	5	6	7	A	B	C	0	1	8	9	10
9	10	0	1	2	A	B	C	6	7	8	3	4	5
5	6	7	8	A	B	C	1	2	3	4	9	10	0
1	2	3	A	B	C	7	8	9	10	0	4	5	6
8	9	A	B	C	2	3	4	5	6	7	10	0	1
4	A	B	C	8	9	10	0	1	2	3	5	6	7
0	5	10	4	9	3	8	2	7	1	6	A	B	C
7	1	6	0	5	10	4	9	3	8	2	B	C	A
3	8	2	7	1	6	0	5	10	4	9	C	A	B

0	1	2	3	4	5	6	7	A	B	C	9	10	8
8	9	10	0	1	2	3	A	B	C	7	5	6	4
5	6	7	8	1	10	A	B	C	3	4	1	2	0
2	3	4	5	6	A	B	C	10	0	1	8	9	7
10	0	1	2	A	B	C	6	7	8	9	4	5	3
7	8	9	A	B	C	2	3	4	5	6	0	1	10
4	5	A	B	C	9	10	0	1	2	3	7	8	6
1	A	B	C	5	6	7	8	9	10	0	3	4	2
A	B	C	1	2	3	4	5	6	7	8	10	0	9
B	C	8	9	10	0	1	2	3	4	A	6	7	5
C	4	5	6	7	8	9	10	0	A	B	2	3	1
6	10	3	7	0	4	8	1	5	9	2	A	B	C
3	7	0	4	8	1	5	9	2	6	10	C	A	B
9	2	6	10	3	7	0	4	8	1	5	B	C	A

4) 15 = 12 + 3, 15 = 11 + 4 are the only decompositions which fulfill the necessary requirements. However, we consider here the latter decomposition since we can utilize the properties of Galois field GF(11).

A	B	C	D	4	5	6	7	8	9	10	0	1	2	3
B	C	D	5	6	7	8	9	10	0	A	1	2	3	4
C	D	6	7	8	9	10	0	1	A	B	2	3	4	5
D	7	8	9	10	0	1	2	A	B	C	3	4	5	6
8	9	10	0	1	2	3	A	B	C	D	4	5	6	0
10	0	1	2	3	4	A	B	C	D	9	5	6	0	1
1	2	3	4	5	A	B	C	D	10	0	6	7	1	2
3	4	5	6	A	B	C	D	0	1	2	7	8	2	3
5	6	7	A	B	C	D	1	2	3	4	8	9	3	4
7	8	A	B	C	D	2	3	4	5	6	9	10	4	5
9	A	B	C	D	3	4	5	6	7	9	10	0	5	6
0	10	9	8	7	6	5	4	3	2	1	A	B	C	D
2	1	0	10	9	8	7	6	5	4	3	B	A	D	C
4	3	2	1	0	10	9	8	7	6	5	C	D	A	B
6	5	4	3	2	1	0	10	9	8	7	D	C	B	A

0	1	2	3	4	5	A	B	C	D	10	8	6	9	7
6	7	8	9	10	A	B	C	D	4	5	2	0	3	1
1	2	3	4	A	B	C	D	9	10	0	7	5	8	6
7	8	9	A	B	C	D	3	4	5	6	1	10	2	0
2	3	A	B	C	D	8	9	10	0	1	6	4	7	5
8	A	B	C	D	2	3	4	5	6	7	0	9	1	10
A	B	C	D	7	8	9	10	0	1	2	5	3	6	4
B	C	D	1	2	3	4	5	6	7	A	10	8	0	9
C	D	6	7	8	9	10	0	1	A	B	4	2	5	3
D	0	1	2	3	4	5	6	A	B	C	9	7	10	8
5	6	7	8	9	10	0	A	B	C	D	3	1	4	2
9	4	10	5	0	6	1	7	2	8	3	A	B	C	D
10	5	0	7	1	7	2	8	3	9	4	D	C	B	A
3	9	4	10	5	0	6	1	7	2	8	B	A	D	C
4	10	5	0	6	1	7	2	8	3	9	C	D	A	B

4) 17 = 13 + 4 and 17 = 12 + 5 are the only decompositions which fulfill the nexessary requirements. The following pair is derived through the first decomposition.

A	B	C	D	4	5	6	7	8	9	10	11	12	0	1	2	3
B	C	D	8	9	10	11	12	0	1	2	3	A	4	5	6	7
C	D	12	0	1	2	3	4	5	6	7	A	B	8	9	10	11
D	3	4	5	6	7	8	9	10	11	A	B	C	12	0	1	2
7	8	9	10	11	12	0	1	2	A	B	C	D	3	4	5	6
12	0	1	2	3	4	5	6	A	B	C	D	11	7	8	9	10
4	5	6	7	8	9	10	A	B	C	D	2	3	11	12	0	1
9	10	11	12	0	1	A	B	C	D	6	7	8	2	3	4	5
1	2	3	4	5	A	B	C	D	10	11	12	0	6	7	8	9
6	7	8	9	A	B	C	D	1	2	3	4	5	10	11	12	0
11	12	0	A	B	C	D	5	6	7	8	9	10	1	2	3	4
3	4	A	B	C	D	9	10	11	12	0	1	2	5	6	7	8
8	A	B	C	D	0	1	2	3	4	5	6	7	9	10	11	12
0	9	5	1	10	6	2	11	7	3	12	8	4	A	B	C	D
5	1	10	6	2	11	7	3	12	8	4	0	9	B	A	D	C
10	6	2	11	7	3	12	8	4	0	9	5	1	C	D	A	B
2	11	7	3	12	8	4	0	9	5	1	10	6	D	C	B	A

0	1	2	3	4	5	6	7	A	B	C	D	12	9	8	11	10
8	9	10	11	12	0	1	A	B	C	D	6	7	3	2	5	4
3	4	5	6	7	8	A	B	C	D	0	1	2	10	9	12	7
11	12	0	1	2	A	B	C	D	7	8	9	10	4	3	6	5
6	7	8	9	A	B	C	D	1	2	3	4	5	11	10	0	12
1	2	3	A	B	C	D	8	9	10	11	12	0	5	4	7	6
9	10	A	B	C	D	2	3	4	5	6	7	8	12	11	1	0
4	A	B	C	D	9	10	11	12	0	1	2	3	6	5	8	7
A	B	C	D	3	4	5	6	7	8	9	10	11	0	12	2	1
B	C	D	10	11	12	0	1	2	3	4	5	A	7	6	9	8
C	D	4	5	6	7	8	9	10	11	12	A	B	1	0	3	2
D	11	12	0	1	2	3	4	5	6	A	B	C	8	7	10	9
5	6	7	8	9	10	11	12	0	A	B	C	D	2	1	4	3
7	0	6	12	5	11	4	10	3	9	2	8	1	A	B	C	D
12	5	11	4	10	3	9	2	8	1	7	0	6	D	C	B	A
10	3	9	2	8	1	7	0	6	12	5	11	4	B	A	D	C
2	8	1	7	0	6	12	5	11	4	10	3	9	C	D	A	B

5) We do not know whether there exists either an 0(14,2) set with 8 common parallel transversals or an 0(15,2) set with 6 common parallel transversals. Therefore the only decomposition of 18 which fulfills the necessary requirements is 18 = 13 + 5. The following pair is constructed through this decomposition.

A	B	C	D	E	5	6	7	8	9	10	11	12	0	1	2	3	4
B	C	D	E	6	7	8	9	10	11	12	0	A	1	2	3	4	5
C	D	E	7	8	9	10	11	12	0	1	A	B	2	3	4	5	6
D	E	8	9	10	11	12	0	1	2	A	B	C	3	4	5	6	7
E	9	10	11	12	0	1	2	3	A	B	C	D	4	5	6	7	8
10	11	12	0	1	2	3	4	A	B	C	D	E	5	6	7	8	9
12	0	1	2	3	4	5	A	B	C	D	E	11	6	7	8	9	10
1	2	3	4	5	6	A	B	C	D	E	12	0	7	8	9	10	11
3	4	5	6	7	A	B	C	D	E	0	1	2	8	9	10	11	12
5	6	7	8	A	B	C	D	E	1	2	3	4	9	10	11	12	0
7	8	9	A	B	C	D	E	2	3	4	5	6	10	11	12	0	1
9	10	A	B	C	D	E	3	4	5	6	7	8	11	12	0	1	2
11	A	B	C	D	E	4	5	6	7	8	9	10	12	0	1	2	3
0	12	11	10	9	8	7	6	5	4	3	2	1	A	B	C	D	E
2	1	0	12	11	10	9	8	7	6	5	4	3	B	C	D	E	A
4	3	2	1	0	12	11	10	9	8	7	6	5	C	D	E	A	B
6	5	4	3	2	1	0	12	11	10	9	8	7	D	E	A	B	C
8	7	6	5	4	3	2	1	0	12	11	10	9	E	A	B	C	D

0	1	2	3	4	5	6	A	B	C	D	E	12	7	9	10	11	8
7	8	9	10	11	12	A	B	C	D	E	5	6	0	2	3	4	1
1	2	3	4	5	A	B	C	D	E	11	12	0	6	8	9	10	7
8	9	10	11	A	B	C	D	E	4	5	6	7	12	1	2	3	0
2	3	4	A	B	C	D	E	10	11	12	0	1	5	7	8	9	6
9	10	A	B	C	D	E	3	4	5	6	7	8	11	0	1	2	12
3	A	B	C	D	E	9	10	11	12	0	1	2	4	6	7	8	5
A	B	C	D	E	2	3	4	5	6	7	8	9	10	12	0	1	11
B	C	D	E	8	9	10	11	12	0	1	2	A	3	5	6	7	4
C	D	E	1	2	3	4	5	6	7	8	A	B	9	11	12	0	10
D	E	7	8	9	10	11	12	0	1	A	B	C	2	4	5	6	3
E	0	1	2	3	4	5	6	7	A	B	C	D	8	10	11	12	9
6	7	8	9	10	11	12	0	A	B	C	D	E	1	3	4	5	2
4	11	5	12	6	0	7	1	8	2	9	3	10	A	B	C	D	E
12	6	0	7	1	8	2	9	3	10	4	11	5	E	A	B	C	D
10	4	11	5	12	6	0	7	1	8	2	9	3	D	E	A	B	C
5	12	6	0	7	1	8	2	9	3	10	4	11	C	D	E	A	B
11	5	12	6	0	7	1	8	2	9	3	10	4	B	C	D	E	A

6) We do not know whether there exists either an $O(18,2)$ set with 8 common parallel transversals or an $O(15,2)$ set with 14 common parallel transversals. Therefore the only decomposition of 22 which fulfill the necessary requirements are $22 = 19 + 3$ and $22 = 17 + 5$.

a: $22 = 19 + 3$,

A	B	C	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	0	1	2
B	C	5	6	7	8	9	10	11	12	13	14	15	16	17	18	0	1	A	2	3	4
C	7	8	9	10	11	12	13	14	15	16	17	18	0	1	2	3	A	B	4	5	6
9	10	11	12	13	14	15	16	17	18	0	1	2	3	4	5	A	B	C	6	7	8
12	13	14	15	16	17	18	0	1	2	3	4	5	6	7	A	B	C	11	8	9	10
15	16	17	18	0	1	2	3	4	5	6	7	8	9	A	B	C	13	14	10	11	12
18	0	1	2	3	4	5	6	7	8	9	10	11	A	B	C	15	16	17	12	13	14
2	3	4	5	6	7	8	9	10	11	12	13	A	B	C	17	18	0	1	14	15	16
5	6	7	8	9	10	11	12	13	14	15	A	B	C	0	1	2	3	4	16	17	18
8	9	10	11	12	13	14	15	16	17	A	B	C	2	3	4	5	6	7	18	0	1
11	12	13	14	15	16	17	18	0	A	B	C	4	5	6	7	8	9	10	1	2	3
14	15	16	17	18	0	1	2	A	B	C	6	7	8	9	10	11	12	13	3	4	5
17	18	0	1	2	3	4	A	B	C	8	9	10	11	12	13	14	15	16	5	6	7
1	2	3	4	5	6	A	B	C	10	11	12	13	14	15	16	17	18	0	7	8	9
4	5	6	7	8	A	B	C	12	13	14	15	16	17	18	0	1	2	3	9	10	11
7	8	9	10	A	B	C	14	15	16	17	18	0	1	2	3	4	5	6	11	12	13
10	11	12	A	B	C	16	17	18	0	1	2	3	4	5	6	7	8	9	13	14	15
13	14	A	B	C	18	0	1	2	3	4	5	6	7	8	9	10	11	12	15	16	17
16	A	B	C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17	18	0
0	17	15	13	11	9	7	5	3	1	18	16	14	12	10	8	6	4	2	A	B	C
3	1	18	16	14	12	10	8	6	4	2	0	17	15	13	11	9	7	5	B	C	A
6	4	2	0	17	15	13	11	9	7	5	3	1	18	16	14	12	10	8	C	A	B

0	1	2	3	4	5	6	7	8	9	10	11	12	A	B	C	16	17	18	14	15	13
13	14	15	16	17	18	0	1	2	3	4	5	A	B	C	9	10	11	12	7	8	6
7	8	9	10	11	12	13	14	15	16	17	A	B	C	2	3	4	5	6	0	1	18
1	2	3	4	5	6	7	8	9	10	A	B	C	14	15	16	17	18	0	12	13	11
14	15	16	17	18	0	1	2	3	A	B	C	7	8	9	10	11	12	13	5	6	4
8	9	10	11	12	13	14	15	A	B	C	0	1	2	3	4	5	6	7	17	18	16
2	3	4	5	6	7	8	A	B	C	12	13	14	15	16	17	18	0	1	10	11	9
15	16	17	18	0	1	A	B	C	5	6	7	8	9	10	11	12	13	14	3	4	2
9	10	11	12	13	A	B	C	17	18	0	1	2	3	4	5	6	7	8	15	16	14
3	4	5	6	A	B	C	10	11	12	13	14	15	16	17	18	0	1	2	8	9	7
16	17	18	A	B	C	3	4	5	6	7	8	9	10	11	12	13	14	15	1	2	0
10	11	A	B	C	15	16	17	18	0	1	2	3	4	5	6	7	8	9	13	14	12
4	A	B	C	8	9	10	11	12	13	14	15	16	17	18	0	1	2	3	6	7	5
A	B	C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18	0	17
B	C	13	14	15	16	17	18	0	1	2	3	4	5	6	7	8	9	A	11	12	10
C	6	7	8	9	10	11	12	13	14	15	16	17	18	0	1	2	A	B	4	5	3
18	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	A	B	C	16	17	15
12	13	14	15	16	17	18	0	1	2	3	4	5	6	7	A	B	C	11	9	10	8
6	7	8	9	10	11	12	13	14	15	16	17	18	0	A	B	C	4	5	2	3	1
11	18	6	13	1	8	15	3	10	17	5	12	0	7	14	2	9	16	4	A	B	C
5	12	0	7	14	2	9	16	4	11	18	6	13	1	8	15	3	10	17	C	A	B
17	5	12	0	7	14	2	9	16	4	11	18	6	13	1	8	15	3	10	B	C	A

b: 22 = 17 + 5,

A	B	C	D	E	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4
B	C	D	E	6	7	8	9	10	11	12	13	14	15	16	0	A	1	2	3	4	5
C	D	E	7	8	9	10	11	12	13	14	15	16	0	1	A	B	2	3	4	5	6
D	E	8	9	10	11	12	13	14	15	16	0	1	2	A	B	C	3	4	5	6	7
E	9	10	11	12	13	14	15	16	0	1	2	3	A	B	C	D	4	5	6	7	8
10	11	12	13	14	15	16	0	1	2	3	4	A	B	C	D	E	5	6	7	8	9
12	13	14	15	16	0	1	2	3	4	5	A	B	C	D	E	11	6	7	8	9	10
14	15	16	0	1	2	3	4	5	6	A	B	C	D	E	12	13	7	8	9	10	11
16	0	1	2	3	4	5	6	7	A	B	C	D	E	13	14	15	8	9	10	11	12
1	2	3	4	5	6	7	8	A	B	C	D	E	14	15	16	0	9	10	11	12	13
3	4	5	6	7	8	9	A	B	C	D	E	15	16	0	1	2	10	11	12	13	14
5	6	7	8	9	10	A	B	C	D	E	16	0	1	2	3	4	11	12	13	14	15
7	8	9	10	11	A	B	C	D	E	0	1	2	3	4	5	6	12	13	14	15	16
9	10	11	12	A	B	C	D	E	1	2	3	4	5	6	7	8	13	14	15	16	0
11	12	13	A	B	C	D	E	2	3	4	5	6	7	8	9	10	14	15	16	0	1
13	14	A	B	C	D	E	3	4	5	6	7	8	9	10	11	12	15	16	0	1	2
15	A	B	C	D	E	4	5	6	7	8	9	10	11	12	13	14	16	0	1	2	3
0	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	A	B	C	D	E
2	1	0	16	15	14	13	12	11	10	9	8	7	6	5	4	3	B	C	D	E	A
4	3	2	1	0	16	15	14	13	12	11	10	9	8	7	6	5	C	D	E	A	B
6	5	4	3	2	1	0	16	15	14	13	12	11	10	9	8	7	D	E	A	B	C
8	7	6	5	4	3	2	1	0	16	15	14	13	12	11	10	9	E	A	B	C	D

0	1	2	3	4	5	6	7	8	A	B	C	D	E	14	15	16	10	11	12	13	9
9	10	11	12	13	14	15	16	A	B	C	D	E	5	6	7	8	1	2	3	4	0
1	2	3	4	5	6	7	A	B	C	D	E	13	14	15	16	0	9	10	11	12	8
10	11	12	13	14	15	A	B	C	D	E	4	5	6	7	8	9	0	1	2	3	16
2	3	4	5	6	A	B	C	D	E	12	13	14	15	16	0	1	8	9	10	11	7
11	12	13	14	A	B	C	D	E	3	4	5	6	7	8	9	10	16	0	1	2	15
3	4	5	A	B	C	D	E	11	12	13	14	15	16	0	1	2	7	8	9	10	6
12	13	A	B	C	D	E	2	3	4	5	6	7	8	9	10	11	15	16	0	1	14
4	A	B	C	D	E	10	11	12	13	14	15	16	0	1	2	3	6	7	8	9	5
A	B	C	D	E	1	2	3	4	5	6	7	8	9	10	11	12	14	15	16	0	13
B	C	D	E	9	10	11	12	13	14	15	16	0	1	2	3	A	5	6	7	8	4
C	D	E	0	1	2	3	4	5	6	7	8	9	10	11	A	B	13	14	15	16	12
D	E	8	9	10	11	12	13	14	15	16	0	1	2	A	B	C	4	5	6	7	3
E	16	0	1	2	3	4	5	6	7	8	9	10	A	B	C	D	12	13	14	15	11
7	8	9	10	11	12	13	14	15	16	0	1	A	B	C	D	E	3	4	5	6	2
16	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	15	11	12	13	14	10
8	9	10	11	12	13	14	15	16	0	A	B	C	D	E	6	7	2	3	4	5	1
5	14	6	15	7	16	8	0	9	1	10	2	11	3	12	4	13	A	B	C	D	E
14	6	15	7	16	8	0	9	1	10	2	11	3	12	4	13	5	E	A	B	C	D
6	15	7	16	8	0	9	1	10	2	11	3	12	4	13	5	14	D	E	A	B	C
15	7	16	8	0	9	1	10	2	11	3	12	4	13	5	14	6	C	D	E	A	B
13	5	14	6	15	7	16	8	0	9	1	10	2	11	3	12	4	B	C	D	E	A

References

Bruck, R. H. (1963). What is a loop? (in Studies in Modern Algebra, ed. A. A. Albert. Prentice Hall, Englewood Cliffs.)

Hall, Marshall, Jr. (1967). Combinatorial Theory. Blaisdell Publishing Company, Massachusetts.

Ryser, J.R. (1963). Combinatorial Mathematics, No. 14 of the Carus Mathematical Monographs, Published by the Amer. Math. Assoc. of Amer., Distributed by John Wiley and Sons, Inc.